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COARSE TOPOLOGIES IN NONSTANDARD EXTENSIONS VIA SEPARATIVE ULTRAFILTERS

BY

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0. Introduction

Let $^*\mathcal{M}$ be a nonstandard extension (ω_1-saturated will do) of a suitably large ground model $\mathcal{M}$. If $A \in \mathcal{M}$ then $^*A$ will denote the image of A in $^*\mathcal{M}$ under the canonical embedding, and $^*[A]$ will denote the set $\{^*a : a \in A\}$. If $\langle X, \tau\rangle$ is a topological space in $\mathcal{M}$ then $^*[\tau]$ is in general no longer a topology but is a basis for what we call the *coarse topology* on $^*X$. This is one of two natural topologies one could put on $^*X$ (the other, generated by $^*\tau$, is the "Q-topology" (see [1], [2], [4], [7], [8]) and is much finer) and is closely related to the "S-topology" (see [6], [8]) used in monad constructions in the setting of uniform spaces.

Our interest here is centered on the question of when $^*X$ (always with the coarse topology) enjoys some of the usual separation properties. As an example, if $\langle \mathbf{R}, v\rangle$ denotes the real line with its usual topology then $^*\mathbf{R}$ can never be a T_0-space when $^*\mathcal{M}$ is ω_1-saturated. In fact, if $^*\mathcal{M}$ is an enlargement (e.g. $^*\mathcal{M}$ is $|\mathcal{M}|^+$-saturated) then $^*X$ is never T_0 for infinite X.

As far as we know, it is an open question whether $^*X$ can be T_0 when X is infinite and $^*\mathcal{M}$ is ω_1-saturated. However, with the help of extra set theory (notably Martin's Axiom (MA) and the Continuum Hypothesis (CH)), we can construct extensions $^*\mathcal{M}$ in which $^*X$ can be T_0 (even Tichonov) for a large class of spaces X.

To begin with, we confine our attention to ultrapower extensions $^*\mathcal{M} = \Pi_D(\mathcal{M})$ where D is a free (that is, nonprincipal) ultrafilter on a countable set I. Then $^*\mathcal{M}$ is automatically ω_1-saturated (since D is countably incomplete) and its elements are equivalence classes $[f] = [f]_D$ of functions $f \in {}^I\mathcal{M}$;

$$[f] = \{g \in {}^I\mathcal{M} : \{i \in I : g(i) = f(i)\} \in D\}.$$

In the case of the nonstandard real line, for example, we have $[f] \mathbin{^*\!<} [g]$ iff $\{i \in I : f(i) < g(i)\} \in D$. For topological spaces $\langle X, \tau\rangle$, if $U \in \tau$ then $^*U = \{[f] : \{i : f(i) \in U\} \in D\}$. (Note that X and $^*[X]$ are naturally homeomorphic ($x \mapsto {}^*x$ is a homeomorphism) and that $^*[X] \subseteq {}^*X$ is a dense subset. This is true for any extension $^*\mathcal{M}$.)

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The ultrafilters of special interest to us for the purposes of separation properties are the so-called "separative" ultrafilters of B. Scott [10] and will be discussed starting in §2. The reader is assumed to be conversant with some of the more well known properties of ultrafilters on a countable set (e.g., selective ultrafilters, P-points, etc.) as well as the Rudin-Keisler order $\leqslant_{RK}$ (see [3], [5], [9]). Our set theoretic notation is standard: $|A|$ is the cardinality of A, ${}^{B}A$ is the set of functions $f : B \to A$, and cardinals are initial ordinals (which are the sets of their ordinal predecessors). Thus $\alpha^{\beta} = |{}^{\beta}\alpha|$ for cardinals α, β. As usual, $\omega = \{0, 1, 2, \ldots\}$, and $c = 2^{\omega}$.

1. General Properties of $^*X$

We will assume always that $^*\mathcal{M}$ is an ω_1-saturated extension (e.g., a countably incomplete ultrapower extension) and that $\langle X, \tau\rangle$ is an infinite topological space. By way of an introductory remark, it is easy to see that $^*[\tau]$ is not in general a topology, even though it is in natural one-one correspondence with τ. Indeed, let D be a free ultrafilter on ω (in terms of the Stone-Čech functor β, $D \in \beta(\omega) \setminus \omega$, where ω has the discrete topology), and let τ be the discrete topology on $X = \omega$. Then

$$\bigcup \{^*\{x\} : x \in X\} = {}^*[X] \notin {}^*[\tau]$$

since $^*[X]$ is countably infinite and members of $^*[\tau]$ are either finite or of cardinal c. (If $A \in \mathcal{M}$ is a countably infinite set then $|^*A| = |\Pi_D(A)| = c$, by a well known property of ultraproducts (see, e.g., [5]).) Thus $^*[\tau]$ is not closed under arbitrary unions. It is also worthy of note that $^*[\sigma]$ needn't basically generate $^*[\tau]$ when σ is a basis for τ. For let $\langle X, \tau\rangle$ be as above and let $\sigma = \{\{x\} : x \in X\}$. Then σ is a basis for the discrete topology; however any union of members of $^*[\sigma]$ will be a subset of $^*[X]$. (The situation is quite different in the case of the Q-topology: $^*\sigma$ is always a basis for $^*\tau$ when σ is a basis for τ.)

Our first proposition is an easy consequence of the definitions involved.

1.1 Proposition. *$^*X \setminus {}^*[X]$ is nonempty and self-dense (i.e., without isolated points). Thus $^*X$ is never discrete.*

The following lemma is true for general ω_1-saturated extensions, and is a basic result of model theory (see, e.g., [5]).

1.2 Lemma. *Let $\langle A_n : n < \omega\rangle$ be a sequence of subsets of X and let $m < \omega$. If $|\bigcap_{n<k} A_n| \geqslant m$ for each $k < \omega$ then $|\bigcap_{n<\omega} {}^*A_n| \geqslant m$.*

1.3 Theorem. *$^*X$ is a nonmetrizable Baire space which is Lindelöf just in case it is compact.*

Proof. This kind of argument has been employed before (see [1], [2], [5]), so we will only sketch it here.

To see that $^*X$ is Baire, let $\langle M_n : n < \omega \rangle$ be a family of dense open subsets of $^*X$, and let $U \in \tau$ be nonempty. We show $^*U \cap (\bigcap_{n<\omega} M_n) \neq \emptyset$. First find nonempty $U_0 \in \tau$ such that $^*U_0 \subseteq {^*U} \cap M_0$. Using induction, we can find nonempty $U_{n+1} \in \tau$ such that $^*U_{n+1} \subseteq {^*U_n} \cap (\bigcap_{k \leq n} M_k)$. By (1.2), $\emptyset \neq \bigcap_{n<\omega} {^*U_n} \subseteq \bigcap_{n<\omega} M_n$.

$^*X$ is nonmetrizable: for if $y \in {^*X} \setminus {^*[X]}$ and if $\langle U_n : n < \omega \rangle \in {^\omega\tau}$ is such that $y \in \bigcap_{n<\omega} {^*U_n}$ then for each $k < \omega$, $\bigcap_{n<k} U_n$ is infinite (since $^*X \setminus {^*[X]}$ is self-dense). Therefore by (1.2), $|\bigcap_{n<\omega} {^*U_n}| \geq 2$; so in fact, $^*X$ fails strongly to be first countable.

Finally, suppose $^*X$ is Lindelöf, and let v be an open cover of $^*X$. We can assume v is countable and consists of basic open sets $\langle {^*U_n} : n < \omega \rangle$. Let $A_n = X \setminus U_n$. Then $\bigcap_{n<\omega} {^*A_n} = \emptyset$, so by (1.2) there is a $k < \omega$ with $\bigcap_{n<k} {^*A_n} = \emptyset$. That is, $\langle {^*U_n} : n < k \rangle$ is a finite subcover of v. ■

A point $x \in X$ is a *weak-P-point* if x is not in the closure of any countable subset of $X \setminus \{x\}$. X is a *weak-P-space* if every point is a weak-P-point, i.e., if all countable subsets are closed. Clearly, a weak-P-space is T_1 and "anticompact" (i.e., no infinite subset is compact); and a P-space which is T_1 is a weak-P-space. We will be concerned with these classes of spaces in the next section; for now we record the following.

1.4 PROPOSITION. *$^*X$ is not a weak-P-space.*

Proof. Let $A \subseteq X$. Then $^*[A]$ is dense in $^*A \subseteq {^*X}$. If $^*[A]$ were closed in $^*X$ then $^*[A]$ would equal $^*A$, whence A would be finite. ■

1.5 PROPOSITION. *If X is compact then $^*X$ is compact but not T_1.*

Proof. Let $\langle {^*U_i} : i \in I \rangle$ be a basic open cover of $^*X$. Then $\langle U_i : i \in I \rangle$ is an open cover of X. If $U_1, \ldots, U_n$ is a finite subcover then $^*X = {^*(U_1 \cup \cdots \cup U_n)} = {^*U_1} \cup \cdots \cup {^*U_n}$, so $^*X$ is compact.

For each $x \in X$, let $\mu(x) = \bigcap\{{^*U} : x \in U \in \tau\}$ denote the "monad" of x. If $^*X$ were T_1 then $\mu(x)$ would be $\{{^*x}\}$ for each $x \in X$; hence given $y \in {^*X} \setminus {^*[X]}$ and $x \in X$ there would be a neighborhood U of x with $y \notin {^*U}$. By compactness, then, there would be a finite subcollection of the U's covering X; consequently $y \notin {^*X}$, an absurdity. ■

1.6 COROLLARY. *If $^*X$ is T_1 then X is anticompact.*

Proof. Suppose $A \subseteq X$ is compact. Then $^*A$ is compact T_1, whence A is finite by (1.5). ■

2. Separative Ultrapower Extensions

For the rest of the paper, we assume $^*\mathcal{M} = \Pi_D(\mathcal{M})$ for some $D \in \beta(I) \setminus I$, I countable (discrete).

2.1 PROPOSITION. *Suppose $^*X$ is T_0. Then D is "separative": for each pair of functions $f, g \in {}^I I$ which are distinct* (*mod D*) (*i.e.*, $\{i \in I : f(i) \neq g(i)\} \in D$) *there is a $J \in D$ such that $f[J] \cap g[J] = \emptyset$.*

Proof. Separative ultrafilters are introduced and studied in [10]. Suppose $f, g \in {}^I I$ are distinct (mod D) and let $h : I \to X$ be one-one. Then $h \circ f$, $h \circ g$ are distinct (mod D). Since $^*X$ is T_0, it is easy to see that there is a set $J \in D$ such that $(h \circ f)[J] \cap (h \circ g)[J] = \emptyset$. Thus $f[J] \cap g[J] = \emptyset$. ∎

2.2 *Remark.* It is straightforward to show that $D \in \beta(I)$ is separative iff whenever $f, g : I \to I$ are distinct (mod D) then their Stone-Čech liftings disagree at D (i.e., $\beta(f)(D) \neq \beta(g)(D)$). (This condition is in fact the definition of separativity used in [10].)

We record the basic facts about separative ultrafilters which will be of use to us here.

2.3 THEOREM (B. Scott [10]). (i) *Selective ultrafilters are separative* (*hence MA implies the existence of separative ultrafilters*).

(ii) *Separativity and being a P-point are not simply related.*

(iii) *If D is separative and $E \leq_{RK} D$ then E too is separative.*

(iv) *If D and E are separative P-points and there is no $F \in \beta(I) \setminus I$ with $F \leq_{RK} D$ and $F \leq_{RK} E$ then*

$$D \cdot E = \{R \subseteq I \times I : \{i : \{j : \langle i, j\rangle \in R\} \in E\} \in D\}$$

is separative (*but not a P-point since $D \cdot E$ is not minimal in the Rudin-Frolík ordering: $D <_{RF} D \cdot E$*).

(v) *$D \cdot D$ is not separative.*

By (2.1, 2.3(v)) we know immediately that $^*X$ is not T_0 whenever X is infinite and $^*\mathcal{M} = \Pi_{D \cdot D}(\mathcal{M})$. Since there is no known proof in ZFC that separative ultrafilters exist, we do not know "absolutely" that coarse topologies can ever have any reasonable separation properties. But, given that D is a separative ultrafilter, quite a lot can be said in this connection.

2.4 PROPOSITION. *If D is separative and X is a Hausdorff P-space then $^*X$ is Hausdorff.*

Proof. Let $[f]$, $[g]$ be distinct and let $J \in D$ be such that $f[J] \cap g[J] = \emptyset$. Since X is a Hausdorff P-space and $f[J]$, $g[J]$ are countable, there are disjoint open sets $U, V \subseteq X$ with $f[J] \subseteq U$, $g[J] \subseteq V$. Thus $[f] \in {}^*U$, $[g] \in {}^*V$, and ${}^*U \cap {}^*V = \emptyset$. ■

Let X be any T_1-space and let $w(X)$ denote the Wallman compactification of X (see [11]). Points of $w(X)$ are ultrafilters of closed subsets of X, and basic open sets are of the form $U^\# = \{p \in w(X) : U$ contains a member of $p\}$ for $U \in \tau$. We identify $x \in X$ with the fixed ultrafilter p_x of closed supersets of $\{x\}$ and define $\varphi : \beta(\omega) \times {}^\omega X \to w(X)$ by

$$\varphi(D, f) = \{A : A \subseteq X \text{ is closed and } f^{-1}[A] \in D\}$$

(easily seen to be a member of $w(X)$).

2.5 Lemma. *Let $\varphi_D : {}^\omega X \to w(X)$ be given by $\varphi_D(f) = \varphi(D, f)$. If X is a weak-P-space and D is a separative ultrafilter then φ_D induces an embedding of ${}^*X$ into $w(X)$ which leaves the points of X fixed (i.e., $\varphi_D({}^*x) = p_x$).*

Proof. Let $f, g : \omega \to X$ be equal (mod D). If $A \in \varphi_D(f)$ then $f^{-1}[A] \in D$. Now $g^{-1}[A] \supseteq f^{-1}[A] \cap \{n : f(n) = g(n)\} \in D$, so $A \in \varphi_D(g)$. Thus φ_D is well defined on ${}^*X$. Let $U \in \tau$. Then $[f] \in \varphi_D^{-1}[U^\#]$ iff there is closed $A \subseteq U$ such that $f^{-1}[A] \in D$ iff there is a closed $A \subseteq U$ such that $[f] \in {}^*A$ iff $[f] \in {}^*U$, since $f[\omega]$ is countable hence closed. Thus φ_D is continuous.

To show $\varphi_D[{}^*U] = \varphi_D[{}^*X] \cap U^\#$, we note that $\varphi_D([f]) \in \varphi_D[{}^*U]$ iff there is a closed $A \subseteq U$ such that $f^{-1}[A] \in D$ iff $\varphi_D([f]) \in U^\#$, again since countable sets are closed.

We need to show φ_D is one-one. Suppose $f, g : \omega \to X$ are distinct (mod D) and let $J \in D$ be such that $f[J] \cap g[J] = \emptyset$. Then $f[J] \in \varphi_D([f])$ and $g[J] \in \varphi_D([g])$, whence these ultrafilters of closed sets are also distinct.

Finally, it is easy to see that points of X are fixed by φ_D, so the proof is complete. ■

2.6 Theorem. *Let D be a separative ultrafilter.*

(i) *If X is a weak-P-space then ${}^*X$ is T_1.*

(ii) *If X is a normal weak-P-space then ${}^*X$ is Tichonov.*

(iii) *If X is a normal P-space then ${}^*X$ is "strongly 0-dimensional" (i.e., disjoint zero sets are separable via clopen sets; equivalently, $\beta(X)$ is "0-dimensional" in the sense of weak inductive dimension).*

(iv) *If X is an extremally disconnected normal weak-P-space then ${}^*X$ is extremally disconnected.*

Proof. (i) By (2.5), ${}^*X$ embeds in $w(X)$, a compact T_1-space.

(ii) If X is normal then $w(X) \simeq \beta(X)$.

(iii) Regular P-spaces are strongly 0-dimensional, hence their Stone-Čech compactifications are 0-dimensional. Now we can make believe that $X \subseteq {}^*X \subseteq \beta(X)$. Thus $\beta({}^*X) \simeq \beta(X)$, whence ${}^*X$ is strongly 0-dimensional.

(iv) $\beta(X)$ is extremally disconnected and ${}^*X$ is a dense subspace. ■

2.7 *Question.* Can $^*X$ ever be Lindelöf T_0?

2.8 THEOREM. *Let X be a normal weak-P-space such that $^*X$ is Lindelöf T_0. Then $|X| > c$.*

Proof. Since $^*X$ is T_0, D is separative. Thus we can consider $X \subseteq {}^*X \subseteq \beta(X)$. By (1.3), $^*X$ is compact, hence equal to $\beta(X)$. Let $A \subseteq X$ be countable discrete. Then A is closed in X, hence C^*-embedded there (see [11]). This says that A is a countable C^*-embedded subset of $\beta(X)$; whence the closure of A in $\beta(X)$ is homeomorphic to $\beta(\omega)$,whose cardinality is well known to be 2^c. Thus $|^*X| \geqslant 2^c$, so $|X| > c$. ■

2.9 *Question.* Is it possible for $^*X$ to be normal? Paracompact?

Motivated by this question, we now turn to the special case of spaces $^*X$ where X is countable discrete ($X = \omega$). First of all notice that by (2.1) and (2.6), D is separative iff $^*\omega$ is T_0 iff $^*\omega$ is an extremally disconnected strongly 0-dimensional space iff $\beta(^*\omega) \simeq \beta(\omega)$. (In [1] it is proved by contrast that topological ultraproducts which are not discrete can never be extremally disconnected unless their cardinalities exceed a measurable cardinal.) A weak affirmative answer to (2.9) is the following.

2.10 THEOREM (CH). *Let D be a separative P-point (e.g., a selective ultrafilter). Then $^*\omega \setminus {}^*[\omega]$ is hereditarily paracompact.*

Proof. We first note that in the embedding $\varphi_D : {}^*\omega \to \beta(\omega)$, the image of φ_D is precisely $\{E \in \beta(\omega) : E \leqslant_{RK} D\}$. (Indeed, $J \in \varphi_D([f])$ iff $f^{-1}[J] \in D$, so $\varphi_D([f]) \leqslant_{RK} D$. On the other hand, if $E \leqslant_{RK} D$ then there is some $f : \omega \to \omega$ such that $J \in E$ iff $f^{-1}[J] \in D$. Hence $E = \varphi_D([f])$.) Thus if D is a P-point as well as being separative then $^*\omega \setminus {}^*[\omega]$ is a P-space. Now $^*\omega$ has an open basis of cardinality $c = \omega_1$, so every subset of $^*\omega \setminus {}^*[\omega]$ is a P-space which is "ω_1-Lindelöf" (i.e., every open cover has a subcover of cardinality less than or equal to ω_1). We are done once we prove the claim (also proved in [1]): If X is an ω_1-Lindelöf regular P-space then every open cover of X refines to an open partition of X. To see this, simply take an open cover v which we can assume to consist of clopen sets and to have cardinality ω_1; say $v = \langle U_\xi : \xi < \omega_1\rangle$. Let $V_\xi = U_\xi \setminus (\cup_{\eta<\xi} U_\eta)$. Then $\langle V_\xi : \xi < \omega_1\rangle$ is an open refinement of v, the members of which are pairwise disjoint. ■

We close this section with a simple observation about covering properties in $^*\omega$ for separative D.

2.11 PROPOSITION. *If D is separative then $^*\omega$ is anticompact, and neither $^*\omega$ nor $^*\omega \setminus {}^*[\omega]$ is Lindelöf.*

Proof. A compact subset of $^*\omega$ is closed in $\beta(\omega)$, and infinite closed subsets of $\beta(\omega)$ are well known to have cardinality 2^c. Since $|{}^*\omega| = c$, no infinite subset can be compact.

Now one of ${}^*\omega$, ${}^*\omega \setminus {}^*[\omega]$ is Lindelöf just in case the other is. If ${}^*\omega$ were Lindelöf, it would, by (1.3), be compact. Impossible. ∎

3. Iterated Ultrapowers

Suppose D, E are free ultrafilters on (countable) sets I, J respectively and let $\mathcal{M}$ be given. Then, letting ${}^{(D)}\mathcal{M}$ denote $\Pi_D(\mathcal{M})$ (to avoid confusion, we replace asterisks with the ultrafilter in brackets) we can iterate the extension process and ask whether ${}^{(D)(E)}\mathcal{M}$ is an ultrapower extension of $\mathcal{M}$. The answer is well known to be "yes"; ${}^{(D)(E)}\mathcal{M}$ is naturally isomorphic (as a membership structure) to ${}^{(D \cdot E)}\mathcal{M}$. The isomorphism is defined as follows. First define $\psi : {}^I({}^J\mathcal{M}) \to {}^{I \times J}\mathcal{M}$ by $\psi(f)(\langle i, j\rangle) = f(i)(j)$. One can then check quite easily that ψ induces an isomorphism

$$\bar{\psi} : {}^{(D)(E)}\mathcal{M} \to {}^{(D \cdot E)}\mathcal{M}, \quad \text{where } \bar{\psi}([f]_D) = [\psi(f)]_{D \cdot E}.$$

Now a natural question to ask is whether $\bar{\psi}$ further induces homeomorphisms between corresponding coarse topologies (as is the case with the Q-topology (see [1])). It is easy to see that, for $U \in \tau$, $\bar{\psi}^{-1}[{}^{(D \cdot E)}U] = {}^{(D)(E)}U$, so $\bar{\psi} \restriction {}^{(D)(E)}X$ is a continuous bijection onto ${}^{(D \cdot E)}X$. In answer to the question of whether $\bar{\psi} \restriction {}^{(D)(E)}X$ is always a homeomorphism, we have the following.

3.1 Proposition. *Let D, E be free ultrafilters on ω. Then $\bar{\psi} \restriction {}^{(D)(E)}\omega$ is not an open map.*

Proof. Since ω has the discrete topology, ${}^{(E)}[\omega] = \bigcup_{n<\omega} {}^{(E)}\{n\}$ is open in ${}^{(E)}\omega$, hence ${}^{(D)(E)}[\omega]$ is open in ${}^{(D)(E)}\omega$. Let $f : \omega \to {}^{(E)}\omega$ be given by $f(m) = {}^{(E)}m$. Then $\{m : f(m) \in {}^{(E)}[\omega]\} = \omega \in D$, so $[f]_D \in {}^{(D)(E)}\omega$. Now $\bar{\psi}([f]_D) = [g]_{D \cdot E}$ where $g(m, n) = m$, and $[h]_{D \cdot E} \in \bar{\psi}[{}^{(D)(E)}[\omega]]$ iff

$$\{m : \{n : h(m, n) = p\} \in E \text{ for some } p\} \in D.$$

If ${}^{(D \cdot E)}J$ is any basic open set containing $[g]_{D \cdot E}$ then $J \in D$, hence J is infinite. Since both D and E are free ultrafilters, we can find $k : \omega \times \omega \to J$ such that $\{m : \{n : k(m, n) = p\} \in E \text{ for some } p\} \notin D$. Thus ${}^{(D \cdot E)}J \not\subseteq \bar{\psi}[{}^{(D)(E)}[\omega]]$, hence $\bar{\psi}[{}^{(D)(E)}[\omega]]$ is not an open set. ∎

3.2 Lemma. *Let D, E be free ultrafilters on ω. Then ${}^{(D)(E)}\omega$ is not a regular space.*

Proof. Look at the proof of (3.1) above. If U is any basic neighborhood of $[f]_D$ which is contained in ${}^{(D)(E)}\omega$ then U must be of the form ${}^{(D)(E)}[J]$ for some $J \in D$. The closure of this set in ${}^{(D)(E)}\omega$ is easily seen to be ${}^{(D)(E)}J$. Again, since both D and E are free, we can find $[g]_D \in {}^{(D)(E)}J$ such that

$\{n : g(n) \text{ is not constant } (\text{mod } E)\} \in D$. Thus ${}^{(D)(E)}[\omega]$ is an open set containing $[f]_D$ which does not contain the closure of any open set containing $[f]_D$. ∎

The following shows that, under CH, ${}^{(D)(E)}X$ and ${}^{(D\cdot E)}X$ can have easily distinguishable topological types.

3.3 THEOREM (CH). *There are ultrafilters D, E on ω such that ${}^{(D\cdot E)}\omega$ is regular, but ${}^{(D)(E)}\omega$ is not regular.*

Proof. Using CH and Theorem (9.13) of [5] there are nonisomorphic selective ultrafilters D, E on ω. Since both are minimal in the Rudin-Keisler ordering, they satisfy the hypothesis of (2.3 (iv)). Thus $D \cdot E$ is separative; so by (2.6), ${}^{(D\cdot E)}\omega$ is a Tichonov space. However, by (3.2), ${}^{(D)(E)}\omega$ fails to be even regular. ∎

3.4 *Remark.* Under the CH, the converses of (2.4) and (2.6 (i)) fail: there is a Hausdorff space X, not a weak-P-space, and an ultrafilter $D \in \beta(\omega) \setminus \omega$ such that ${}^{(D)}X$ is Hausdorff. For let D, E be as in (3.3), and let $X = {}^{(E)}\omega$. Since ${}^{(D\cdot E)}\omega$ is Tichonov, hence Hausdorff, and the natural bijection $\overline{\psi} \restriction {}^{(D)(E)}\omega$ is continuous, we know that ${}^{(D)}X$ is also Hausdorff. But X fails to be a weak-P-space by (1.4).

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